JOURNAL OF DIFFERENTIAL EQUATIONS 9, 239–252 (1971)

An Invariance Principle for Compact Processes

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Received January 19, 1970

1. INTRODUCTION

The classical Liapunov theory of asymptotic stability relies on the construction of a Liapunov functional with a negative definite time derivative. Nevertheless, for a wide class of evolutionary equations of mathematical and physical interest, the inherent dissipative mechanism yields "natural" Liapunov functionals with negative semidefinite time derivatives.

It was pointed out by LaSalle [1] that in the case of autonomous systems of ordinary differential equations the negative definiteness condition can be weakened if one takes into account the invariance of limit sets of solutions.[1] An asymptotic stability theory which utilizes the invariance principle is now available for general topological dynamical systems (Hale [3]).

In view of the success of this approach, a search was conducted for uncovering cases of nonautonomous systems of ordinary differential equations endowed with an invariance principle. Generalized forms of invariance principles were established for the classes of asymptotically autonomous (Markus [4], Opial [5]), periodic (LaSalle [6]) and asymptotically almost periodic systems (Miller [7]). Sell [8] introduces the concept of "limiting equations" and deduces the above results in a systematic way.

Invariance principles have also been considered for other types of evolutionary equations which do not generate dynamical systems. In this category we mention Miller's work [9] on almost periodic functional differential equations and a forthcoming article [10] by Slemrod on periodic dynamical systems in Banach spaces.

The conceptual similarity between results concerning a wide diversity of evolutionary equations motivates the development of an abstract unifying theory. In this work we study invariance in the framework of the theory of "processes." A process is a direct generalization of the concept of a topological dynamical system. The transition from dynamical systems to

[1] See also Barbashin and Krasovskii [2].

processes is, roughly speaking, analogous to the transition from autonomous to nonautonomous systems of ordinary differential equations.

Of a particular interest to us is the class of "compact processes" which is introduced in Section 2. In Section 3 it is shown that compact processes are endowed with an invariance principle. For compact processes generated by systems of ordinary differential equations the invariance principle is very similar to that given in [8]. Results of this character are deduced in Section 4 where it is established that some interesting types of processes are compact. In particular, topological dynamical systems are compact processes and the results of [3] are recovered as special cases.

In the final Section 5 we introduce the concept of a Liapunov functional for a compact process. We then proceed to demonstrate how the invariance principle can be combined with the existence of a Liapunov functional to yield an asymptotic stability theorem.

Applications of the theory to specific evolutionary systems of functional equations will be exhibited in a forthcoming paper [12].

2. Compact Processes

Throughout this paper X will denote a metric space with metric d. We will be using the standard symbols R and R^+ for the sets of real and real non-negative numbers, respectively.

We begin with some basic definitions:

Definition 2.1. *A process* on X is a mapping

$$u : R \times X \times R^+ \to X$$

with the following properties:

(i) $u^t(x, 0) = x \quad$ for all $\quad t \in R, x \in X, \qquad (2.1)$

(ii) $u^t(x, s + \tau) = u^{t+s}(u^t(x, s), \tau)$ for all $t \in R$, $x \in X$, $\tau, s \in R^+, \qquad (2.2)$

(iii) For fixed $\tau \in R^+$, the one parameter family of maps

$$u^t(\cdot, \tau) : X \to X, \quad \text{parameter} \quad t \in R,$$

is equicontinuous.

Remark 2.1. Note that every process on X can be extended by continuity onto a process on the completion of X. Thus, from now on and without loss of generality we will assume that X is complete.

DEFINITION 2.2. Let u be a process on X and (t, x) a point in $R \times X$. The orbit of u which originates at (t, x) is the map

$$u^t(x, \cdot) : R^+ \to X.$$

It is clear that solutions of well-posed initial value problems of an evolutionary functional equation can be visualized as orbits of an appropriate process. In such a case $u^t(x, \tau)$ will give the solution at time $t + \tau$ of the problem which assigns at time t initial data x.

DEFINITION 2.3. A process u on X is called *continuous* (resp. *uniformly continuous*) if for any fixed $x \in X$, $\tau \in R^+$, the map

$$u^{\cdot}(x, \tau) : R \to X$$

is continuous (resp. uniformly continuous) on R.

Remark 2.2. Recalling requirements (i), (ii), (iii) of Definition 2.1 and using the triangle inequality

$$d(u^{t+s}(x, \tau), u^t(x, \tau)) \leqslant d(u^{t+s}(x, \tau), u^{t+s}(u^t(x, s), \tau)) + d(u^t(x, \tau + s), u^t(x, \tau)) \tag{2.3}$$

one deduces that if the orbits of a process u are continuous, then u is continuous. Assuming, further, that for any fixed $x \in X$ the orbits $u^t(x, \cdot)$ form an equicontinuous family with parameter $t \in R$, it follows that u is uniformly continuous.

DEFINITION 2.4. Let u be a process on X and $T \in R$. The *T-translate* of u is a process u_T on X defined by

$$u_T{}^t(x, \tau) \equiv u^{t+T}(x, \tau) \quad \text{for all} \quad t \in R, \quad x \in X, \quad \tau \in R^+. \tag{2.4}$$

We are now ready to introduce the concept of a compact process.

DEFINITION 2.5. A process u on X is called *compact* if the set of right translates $\{u_T \mid T \in R^+\}$ is sequentially conditionally compact relative to the pointwise topology, i.e., for any sequence $\{T_n\}$ in R^+ there exists a subsequence $\{T_{n_k}\}$ and a map $v : R \times X \times R^+ \to X$ such that

$$u^t_{T_{n_k}}(x, \tau) \to v^t(x, \tau), \qquad k \to \infty, \qquad \text{for all} \quad t \in R, \ x \in X, \ \tau \in R^+.$$

DEFINITION 2.6. A process u on X is called *uniformly compact* (resp. *uniformly compact to the right*) if for any sequence $\{T_n\}$ in R^+ there exists a subsequence $\{T_{n_k}\}$ and a map v: $R \times X \times R^+ \to X$ such that

$$u^t_{T_{n_k}}(x, \tau) \to v^t(x, \tau), \qquad k \to \infty, \qquad \text{for all} \quad t \in R, \; x \in X, \; \tau \in R^+,$$

the convergence being uniform in t on R (resp. on every subset of R which is bounded from below).

Remark 2.3. Suppose that X is separable and u is a process on X whose orbits $u^t(x, \cdot)$ form, for any fixed $x \in X$, an equicontinuous family with parameter $t \in R$. In this case we have the following simple criterion of compactness: u is compact if and only if, for any fixed $x \in X$, $\tau \in R^+$, the set $\{u^t(x, \tau) \mid t \in R^+\}$ is precompact in X.

DEFINITION 2.7. Let u be a compact process on X. The *asymptotic hull* $H[u]$ of u is defined as the set of all maps v: $R \times X \times R^+ \to X$ with the property that there exists a sequence $\{T_n\}$ in R^+, $T_n \to \infty$ as $n \to \infty$, such that $u_{T_n} \to v$, $n \to \infty$, in the pointwise topology.

The structure of the asymptotic hull of a compact process is illuminated by the following

PROPOSITION 2.1. *Let u be a compact process on X with asymptotic hull $H[u]$. If $v \in H[u]$, then v is a process on X.*

Proof. We have to prove that the three requirements of Definition 2.1 are satisfied, namely,

$$v^t(x, 0) = x \qquad \text{for all} \quad t \in R, \quad x \in X, \tag{2.5}$$

$$v^t(x, s + \tau) = v^{t+s}(v^t(x, s), \tau) \qquad \text{for all} \quad t \in R, \quad x \in X, \quad s, \tau \in R^+, \tag{2.6}$$

for any fixed $x \in X$, $\tau \in R^+$, $\epsilon > 0$, there exists $\delta = \delta(x, \tau, \epsilon) > 0$ such that

$$d(v^t(y, \tau), v^t(x, \tau)) \leqslant \epsilon \quad \text{if} \quad d(y, x) \leqslant \delta, \quad \text{for all} \quad t \in R. \tag{2.7}$$

Since $v \in H[u]$, there exists $\{T_n\}$ in R^+, $T_n \to \infty$ as $n \to \infty$, such that

$$u_{T_n} \to v, \qquad n \to \infty, \tag{2.8}$$

in the pointwise topology.

Equation (2.5) is an immediate consequence of (2.8), (2.4), and (2.1).

To prove (2.6), fix $t \in R$, $x \in X$, $s, \tau \in R^+$. Using (2.2), (2.4) and the triangle inequality, we obtain for $n = 1, 2, \ldots,$

$$d(v^t(x, s + \tau), v^{t+s}(v^t(x, s), \tau)) \leqslant d(v^t(x, s + \tau), u^t_{T_n}(x, s + \tau))$$
$$+ d(u^{t+s+T_n}(u^t_{T_n}(x, s), \tau), u^{t+s+T_n}(v^t(x, s), \tau))$$
$$+ d(u^{t+s}_{T_n}(v^t(x, s), \tau), v^{t+s}(v^t(x, s), \tau)).$$

On account of (2.8) and requirement (iii) of Definition 2.1, the right hand side of the above inequality tends to zero as $n \to \infty$ and (2.6) is established.

Since u is a process, for fixed $x \in X$, $\tau \in R^+$ and any $\epsilon > 0$ there exists $\delta = \delta(x, \tau, \epsilon) > 0$ such that

$$d(u^s(y, \tau), u^s(x, \tau)) \leqslant \epsilon \qquad \text{if} \quad d(y, x) \leqslant \delta \qquad \text{for all} \quad s \in R. \tag{2.9}$$

For any $t \in R$, $x, y \in X$, $\tau \in R^+$ and $n = 1, 2, \ldots,$

$$d(v^t(y, \tau), v^t(x, \tau)) \leqslant d(v^t(y, \tau), u^t_{T_n}(y, \tau))$$
$$+ d(u^{t+T_n}(y, \tau), u^{t+T_n}(x, \tau)) + d(u^t_{T_n}(x, \tau), v^t(x, \tau)).$$

From (2.8) and (2.9) we deduce that the infimum of the right hand side of the above inequality does not exceed ϵ and (2.7) follows. Q.E.D.

DEFINITION 2.8. A *limiting process* of a compact process u is a member of the asymptotic hull of u.

Note that there is no guarantee that the limiting processes of a compact process are compact. However, we have the following

PROPOSITION 2.2. *Let v be a limiting process of a process u which is uniformly compact to the right. Then v is uniformly compact to the right and $H[v] = H[u]$.*

Proof. There exists $\{T_n\}$ in R^+, $T_n \to \infty$ as $n \to \infty$, such that

$$u^t_{T_n}(x, \tau) \to v^t(x, \tau), \qquad n \to \infty, \qquad \text{for all} \quad t \in R, \ x \in X, \ \tau \in R^+, \tag{2.10}$$

the convergence being uniform in t on every subset of R which is bounded from below.

Let $\{S_n\}$ be any sequence in R^+. We set $Q_n \equiv T_n + S_n$. Since u is uniformly compact to the right, there exists a subsequence $\{Q_{n_k}\}$ of $\{Q_n\}$ as well as $w \in H[u]$ with the property

$$u^t_{Q_{n_k}}(x, \tau) \to w^t(x, \tau), \qquad k \to \infty, \qquad \text{for all} \quad t \in R,\ x \in X,\ \tau \in R^+, \tag{2.11}$$

uniformly in t on every subset of R which is bounded from below. For any fixed $t \in R$, $x \in X$, $\tau \in R^+$,

$$d(v^t_{S_{n_k}}(x, \tau), w^t(x, \tau)) \leqslant d(v^{t+S_{n_k}}(x, \tau), u^{t+S_{n_k}}_{T_{n_k}}(x, \tau)) + d(u^t_{Q_{n_k}}(x, \tau), w^t(x, \tau)).$$

Using (2.10), (2.11), and the above inequality we conclude

$$v^t_{S_{n_k}}(x, \tau) \to w^t(x, \tau), \qquad k \to \infty, \qquad \text{for all} \quad t \in R,\ x \in X,\ \tau \in R^+,$$

the convergence being uniform in t on every subset of R which is bounded from below, i.e., v is uniformly compact to the right.

To prove that $H[u] = H[v]$, assume first that $w \in H[v]$. There exists $\{S_n\}$ in R^+, $S_n \to \infty$ as $n \to \infty$, such that $v_{S_n} \to w$, $n \to \infty$. We set $Q_n \equiv T_n + S_n$ and we use the inequality

$$d(u^t_{Q_n}(x, \tau), w^t(x, \tau)) \leqslant d(u^{t+S_n}_{T_n}(x, \tau), v^{t+S_n}(x, \tau)) + d(v^t_{S_n}(x, \tau), w^t(x, \tau)),$$

whose right hand side tends to zero as $n \to \infty$, to deduce that $w \in H[u]$.

Assume now that $w \in H[u]$. There exists sequence $\{Q_k\}$ in R^+, $Q_k \to \infty$ as $k \to \infty$, such that $u_{Q_k} \to w$, $k \to \infty$. We select a subsequence $\{Q_{k_n}\}$ so that $Q_{k_n} \geqslant T_n + n$ and we set $S_n \equiv Q_{k_n} - T_n$. Obviously $S_n \to \infty$ as $n \to \infty$. The right hand side of the inequality

$$d(v^t_{S_n}(x, \tau), w^t(x, \tau)) \leqslant d(v^{t+S_n}(x, \tau), u^{t+S_n}_{T_n}(x, \tau)) + d(u^t_{Q_{k_n}}(x, \tau), w^t(x, \tau))$$

tends to zero as $n \to \infty$. Therefore, $v_{S_n} \to w$ as $n \to \infty$ and $w \in H[v]$. Q.E.D.

The following property of the asymptotic hull of a compact process is of some interest:

PROPOSITION 2.3. *The asymptotic hull $H[u]$ of a compact process u is translationally invariant, i.e., $v \in H[u]$ implies $v_T \in H[u]$ for any $T \in R$.*

Proof. Let $v \in H[u]$. There exists a sequence $\{T_n\}$ in R^+ such that $u_{T_n} \to v$, $n \to \infty$, in the pointwise topology. Fix $T \in R$. For any $t \in R$, $x \in X$, $\tau \in R^+$,

$$\lim_{n\to\infty} u^t_{T+T_n}(x, \tau) = \lim_{n\to\infty} u^{t+T}_{T_n}(x, \tau) = v^{t+T}(x, \tau) = v_T{}^t(x, \tau),$$

which shows that $v_T \in H[u]$. Q.E.D.

3. The Invariance Principle

From the theory of dynamical systems we transplant here the following definitions:

Definition 3.1. Let u be a process on X and (t, x) a point in $R \times X$. The *ω-limit set* of the orbit of u which originates at (t, x) is defined by

$$\omega^t(x) \equiv \bigcap_{s \geqslant 0} Cl_X \bigcup_{\tau \geqslant s} u^t(x, \tau).$$

Definition 3.2. Let u be a compact process on X. A subset Y of X is called ***positively invariant*** relative to u if for each fixed $t \in R$ and each $y \in Y$ there exists a limiting process $v \in H[u]$ such that $v^t(y, \tau) \in Y$ for all $\tau \in R^+$.

Remark 3.1. On account of Proposition 2.3, the statement "for each fixed $t \in R$ there exists $v \in H[u]$" is equivalent to the weaker "for some $t \in R$ there exists $v \in H[u]$".

Proposition 3.1. (The Weak Invariance Principle). *Let u be a compact process on X. Suppose that the ω-limit set $\omega^t(x)$ of the orbit of u which originates at some point (t, x) is not empty. Then $\omega^t(x)$ is positively invariant relative to u.*

Proof. Let $y \in \omega^t(x)$. There exists a sequence $\{T_n\}$ in R^+, $T_n \to \infty$ as $n \to \infty$, such that

$$u^t(x, T_n) \to y, \qquad n \to \infty. \tag{3.1}$$

Since u is compact, there exists a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and a limiting process $v \in H[u]$ with the property

$$u_{T_{n_k}} \to v, \qquad k \to \infty, \qquad \text{in the pointwise topology.} \tag{3.2}$$

For any fixed $\tau \in R^+$ and on account of (2.2),

$$\begin{aligned} d(u^t(x, \tau + T_{n_k}), v^t(y, \tau)) \leqslant\ & d(u^{t+T_{n_k}}(u^t(x, T_{n_k}), \tau), u^{t+T_{n_k}}(y, \tau)) \\ & + d(u^t_{T_{n_k}}(y, \tau), v^t(y, \tau)). \end{aligned} \tag{3.3}$$

Recalling (3.1) and (3.2) we conclude that as $k \to \infty$ the right hand side of the above inequality converges to zero. Thus $v^t(y, \tau) \in \omega^t(x)$ and invariance is established with the help of Remark 3.1. Q.E.D.

The information provided by the weak invariance principle is particularly valuable if the range of the orbit is precompact in X. In fact in this case one can prove easily the following.

PROPOSITION 3.2. *Let u be a process on X. Suppose that the range of the orbit of u which originates at some point (t, x) is precompact in X. Then the ω-limit set $\omega^t(x)$ of the orbit is nonempty and compact. Furthermore,*

$$u^t(x, \tau) \to \omega^t(x), \ \tau \to \infty. \tag{3.4}$$

A stronger invariance principle for the ω-limit set of an orbit of a compact process can be established if the orbit satisfies appropriate smoothness assumptions (cf. Hale [3]).

DEFINITION 3.3. Let u be a compact process on X. A subset Y of X is called *invariant* relative to u if there exists a map $\bar{v} : Y \times R \to Y$ with the following properties:

(i) $\bar{v}(y, 0) = y$ for all $y \in Y$.

(ii) For each $y \in Y$ there is a limiting process $v \in H[u]$ such that

$$\bar{v}(y, s + \tau) = v^{t+s}(\bar{v}(y, s), \tau) \text{ for all } s \in R,\ \tau \in R^+ \text{ and some } t \in R. \tag{3.5}$$

Remark 3.2. In particular, (3.5) for $s = 0$ implies that the restriction of $\bar{v}$ on $y \times R^+$ coincides with the restriction of v on $t \times y \times R^+$. Therefore, invariance implies positive invariance.

PROPOSITION 3.3. (The Invariance Principle). *Let u be a compact process on X. Suppose that the orbit of u which originates at some point (t, x) is uniformly continuous on R^+ and its range is precompact in X. Then the ω-limit set $\omega^t(x)$ of the orbit is nonempty, compact and invariant relative to u.*

Proof. (Compare with the proof of Lemma 3 in [3]). The set $\omega^t(x)$ is nonempty and compact in virtue of Proposition 3.2. Let $y \in \omega^t(x)$. There exists a sequence $\{T_n\}$ in R^+, $T_n \to \infty$ as $n \to \infty$, such that

$$u^t(x, T_n) \to y, \qquad n \to \infty. \tag{3.6}$$

We define

$$u_n(x, \tau) \equiv \begin{cases} u^t(x, \tau + T_n), & \tau \in [-T_n, \infty) \\ u^t(x, -\tau - T_n), & \tau \in (-\infty, -T_n) \end{cases} \qquad n = 1, 2, \ldots . \tag{3.7}$$

Since $u^t(x, \cdot)$ is uniformly continuous on R^+, $\{u_n(x, \cdot)\}$ is uniformly equicontinuous on R. Furthermore, for any fixed $\tau \in R$, $\{u_n(x, \tau)\}$ is precompact in X. We now employ the Ascoli–Arzela theorem to deduce that $\{u_n(x, \cdot)\}$ is conditionally compact relative to the topology of uniform convergence on compacta, i.e., there exists a subsequence $\{u_{n_k}(x, \cdot)\}$ and a function denoted by $\bar{v}(y, \cdot)$ such that

$$u_{n_k}(x, \tau) \to \bar{v}(y, \tau), \quad k \to \infty, \quad \text{for all } \tau \in R, \quad \text{uniformly on compacta.} \tag{3.8}$$

On account of (3.8) and (3.7), $\bar{v}(y, \tau) \in \omega^t(x)$ for any $\tau \in R$. Moreover, using (3.8), (3.7), and (3.6),

$$\bar{v}(y, 0) = \lim_{k\to\infty} u_{n_k}(x, 0) = \lim_{k\to\infty} u^t(x, T_{n_k}) = y.$$

Since u is compact, there is a subsequence of $\{T_{n_k}\}$, which for simplicity is again labeled by $\{T_{n_k}\}$, and a limiting process $v \in H[u]$ with the property

$$u_{T_{n_k}} \to v, \qquad k \to \infty, \qquad \text{in the pointwise topology.} \tag{3.9}$$

Fix $s \in R$, $\tau \in R^+$. For sufficiently large k, so that $T_{n_k} \geqslant s$,

$$\begin{aligned} d(v^{t+s}(\bar{v}(y, s), \tau), \bar{v}(y, s + \tau)) \leqslant\ & d(v^{t+s}(\bar{v}(y, s), \tau), u_{T_{n_k}}^{t+s}(\bar{v}(y, s), \tau)) \\ & + d(u^{t+s+T_{n_k}}(\bar{v}(y, s), \tau), u^{t+s+T_{n_k}}(u_{n_k}(x, s), \tau)) \\ & + d(u_{n_k}(x, s + \tau), \bar{v}(y, s + \tau)), \end{aligned}$$

where use has been made of the identity

$$u^{t+s+T_{n_k}}(u_{n_k}(x, s), \tau) = u_{n_k}(x, s + \tau)$$

which can be established with the help of (2.2) and (3.7). In virtue of (3.9), (3.8), and requirement (iii) of Definition 2.1, the right hand side of the above inequality tends to zero as $k \to \infty$. It follows that the left hand side vanishes and (3.5) is satisfied. Q.E.D.

4. Some Special Classes of Compact Processes

In this section we exhibit some interesting types of processes which are compact.

Definition 4.1. A *dynamical system* on X is a process u on X such that $u_T = u$ for any $T \in R$.

It is obvious that every dynamical system u is a compact process with asymptotic hull $H[u] = \{u\}$.

Remark 4.1. In the literature (e.g., Hale [3]), the requirement of continuity on $X \times R^+$ is embodied in the definition of a dynamical system.[2] Motivated by an example which arose in viscoelasticity (Dafermos [11]), we have relaxed here this restriction. Note that in establishing the invariance principle (Proposition 3.3) we do require that the test orbit is uniformly continuous but we make no assumption regarding the continuity of the remaining orbits of the process.

DEFINITION 4.2. A process u on X is called *periodic* with period $T > 0$ if $u_T = u$.

PROPOSITION 4.1. *A continuous periodic process is uniformly compact and its asymptotic hull is identical to the set of translates.*

Proof. Let u a continuous periodic process on X with period T. Note that periodicity together with continuity imply that u is uniformly continuous. For any sequence $\{T_n\}$ in R^+ we write $T_n = \lambda_n T + \tau_n$ where λ_n is an integer and $\tau_n \in [0, T)$. There is $s \in [0, T]$ and a subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ such that

$$\tau_{n_k} \to s, \qquad k \to \infty. \tag{4.1}$$

Using (4.1), the periodicity and the uniform continuity of u, we obtain for any fixed $t \in R$, $x \in X$, $\tau \in R^+$,

$$\lim_{k\to\infty} u_{T_{n_k}}^t(x, \tau) = \lim_{k\to\infty} u_{\tau_{n_k}}^t(x, \tau) = \lim_{k\to\infty} u^{t+\tau_{n_k}}(x, \tau) = u^{t+s}(x, \tau) = u_s{}^t(x, \tau),$$

the convergence being uniform in t on R. It follows that u is uniformly compact and $H[u]$ is identical to the set of s-translates of u, $s \in [0, T]$, or, equivalently (by periodicity), to the set of all translates of u. Q.E.D.

Remark 4.2. Let u be a periodic process which is continuous on $R \times X \times R^+$. It can be shown that every orbit of u with precompact range is uniformly continuous on R^+ and the Invariance Principle (Proposition 3.3) reduces to its form in Slemrod [10].

DEFINITION 4.3. A process on X is called *almost periodic* if it is uniformly compact.

Proposition 4.1 implies that continuous periodic processes are almost

[2] This implies in turn that every orbit with precompact range is uniformly continuous on R^+ and the Invariance Principle (Proposition 3.3) reduces to its form in Hale [3].

periodic. The name "almost periodic" in Definition 4.3 has been motivated by the following proposition which is an adaptation of a classical theorem of Bochner.

PROPOSITION 4.2. *Let u be an almost periodic process on X. For fixed $x \in X$, $\tau \in R^+$ and any $\epsilon > 0$ there exists a set Γ_ϵ, relatively dense in R, such that if $T_\epsilon \in \Gamma_\epsilon$, then*

$$d(u^t_{T_\epsilon}(x, \tau), u^t(x, \tau)) \leqslant \epsilon, \qquad \textit{for all} \quad t \in R. \tag{4.2}$$

Proof. Because of uniform compactness, for any $\epsilon > 0$, there exists a finite set $\{T_1, ..., T_m\}$ in R with the following property: Given $T \in R$ there is an index i, $1 \leqslant i \leqslant m$, such that

$$d(u_T{}^t(x, \tau), u^t_{T_i}(x, \tau)) \leqslant \epsilon, \qquad \text{for all} \quad t \in R,$$

whence

$$d(u^t_{T-T_i}(x, \tau), u^t(x, \tau)) \leqslant \epsilon, \qquad \text{for all} \quad t \in R.$$

In other words for any $T \in R$ there exists T_i such that $T - T_i \in \Gamma_\epsilon$. If we set $L_\epsilon = \max_{1 \leqslant i \leqslant m} | T_i |$ we conclude that every closed interval of length L_ϵ contains at least one point of Γ_ϵ, i.e. Γ_ϵ is relatively dense. Q.E.D.

Another interesting class of compact processes is the following:

DEFINITION 4.4. Two compact processes on X will be called *asymptotically equivalent* if their asymptotic hulls coincide. A compact process on X which is asymptotically equivalent to a dynamical system, a periodic process or an almost periodic process will be called *asymptotically a dynamical system*, *asymptotically periodic* or *asymptotically almost periodic*, respectively.

We observe that a compact process is asymptotically a dynamical system if and only if its asymptotic hull contains a single element.

5. LIAPUNOV FUNCTIONALS FOR COMPACT PROCESSES

In the theory of dynamical systems the existence of a Liapunov functional combined with the invariance principle yields powerful asymptotic stability theorems. Here we generalize the procedure in the framework of the theory of compact processes.

Definition 5.1. Let u be a compact process on X. A map

$$V : R \times X \to R$$

is called a Liapunov functional for u if the following conditions are satisfied:

(i) The one parameter family of maps

$$V^t(\cdot) : X \to R, \qquad t \in R,$$

is equicontinuous

(ii) For any fixed $t \in R$,

$$\dot{V}^t(x) \equiv \lim_{s \to 0+} \sup \frac{1}{s} [V^{t+s}(u^t(x, s)) - V^t(x)] \leqslant 0. \tag{5.1}$$

(iii) If $\{T_n\}$ is a sequence in R^+ such that $\{u_{T_n}\}$ is convergent in the pointwise topology, then the sequence $\{V^{t+T_n}(x)\}$ is also convergent for any $t \in R$, $x \in X$.[3]

Definition 5.2. Let u be a compact process on X with asymptotic hull $H[u]$. Suppose that V is a Liapunov functional for u. The *limiting Liapunov functional generated by* V is a map

$$W : H[u] \times R \times X \to R$$

constructed in the following way: For $v \in H[u]$, $t \in R$, $x \in X$,

$$W_v{}^t(x) \equiv \lim_{n \to \infty} V^{t+T_n}(x) \tag{5.2}$$

where $\{T_n\}$ is any sequence in R^+ such that $u_{T_n} \to v$ as $n \to \infty$.

The following theorem supplements Proposition 3.1 when a Liapunov functional exists.

Proposition 5.1. *Let u be a compact process on X equipped with a Liapunov functional V which generates a limiting Liapunov functional W. Suppose that the ω-limit set $\omega^t(x)$ of the orbit of u which originates at some point (t, x) is not empty. Then, given $y \in \omega^t(x)$, there exists a limiting process $v \in H[u]$ such that $v^t(y, \tau) \in \omega^t(x)$ for all $\tau \in R^+$ and*

$$\dot{W}_v{}^t(y) \equiv \lim_{s \to 0+} \sup \frac{1}{s} [W_v^{t+s}(v^t(y, s)) - W_v{}^t(y)] = 0. \tag{5.3}$$

[3] Suppose that u is continuous periodic. Then this condition will be satisfied if and only if $V^t(x)$ is continuous periodic in t and every period of u is also a period of V.

Proof. Recalling the proof of Proposition 3.1 and in particular (3.2), (3.3), we deduce that there exists $v \in H[u]$ and a sequence $\{T_n\}$ in R^+, $T_n \to \infty$ as $n \to \infty$, such that

$$u_{T_n} \to v, \qquad n \to \infty, \qquad \text{in the pointwise topology,} \tag{5.4}$$

$$u^t(x, \tau + T_n) \to v^t(y, \tau), \qquad n \to \infty, \qquad \text{for all} \quad \tau \in R^+, \tag{5.5}$$

which implies that $v^t(y, \tau) \in \omega^t(x)$, $\tau \in R^+$.

For any $\tau \in R^+$ and $n = 1, 2,\ldots,$

$$\begin{aligned} &d(V^{t+\tau+T_n}(u^t(x, \tau + T_n)), W_v^{t+\tau}(v^t(y, \tau))) \\ &\qquad \leqslant d(V^{t+\tau+T_n}(u^t(x, \tau + T_n)), V^{t+\tau+T_n}(v^t(y, s))) \\ &\qquad\quad + d(V^{t+\tau+T_n}(v^t(y, s)), W_v^{t+\tau}(v^t(y, \tau))). \end{aligned}$$

On account of (5.5) and (5.2), the right hand side of the above inequality tends to zero as $n \to \infty$ whence

$$\lim_{n\to\infty} V^{t+\tau+T_n}(u^t(x, \tau + T_n)) = W_v^{t+\tau}(v^t(y, \tau)), \qquad \text{for all} \quad \tau \in R^+. \tag{5.6}$$

By (5.1), $V^{t+s}(u^t(x, s))$ is a decreasing function of s on R^+. Hence there is $V_\infty \geqslant -\infty$ with the property

$$V^{t+s}(u^t(x, s)) \to V_\infty, \qquad s \to \infty. \tag{5.7}$$

Combining (5.6) with (5.7) we conclude that V_∞ is finite and

$$W_v^{t+\tau}(v^t(y, \tau)) = V_\infty \qquad \text{for all} \quad \tau \in R^+. \tag{5.8}$$

From (5.8) it follows that $\dot{W}_v{}^t(y) = 0$. Q.E.D.

Proposition 5.1 can be exploited in the study of asymptotic stability for an orbit whose range is precompact in X. In fact in this case (3.4) demonstrates that the ω-limit set characterizes completely the asymptotic behavior of the orbit. If the orbit is also uniformly continuous on R^+, more detailed information on the asymptotics can be obtained by combining the assertions of Propositions 3.3 and 5.1. Namely,

PROPOSITION 5.2. *Let u be a compact process on X equipped with a Liapunov functional V which generates a limiting Liapunov functional W. Suppose that the orbit of u which originates at some point (t, x) is uniformly continuous in R^+*

and its range is precompact in X. Then the ω-limit set $\omega^t(x)$ of the orbit is nonempty and compact. Furthermore, there exists a map $\bar{v} : \omega^t(x) \times R \to \omega^t(x)$ with the following properties:

$$\text{(i)} \quad \bar{v}(y, 0) = y \qquad \textit{for all} \quad y \in \omega^t(x). \tag{5.9}$$

(ii) *For any $y \in \omega^t(x)$ there is a limiting process $v \in H[u]$ so that* (3.5) *is satisfied and*

$$W_v^{t+s}(\bar{v}(y, s)) = W_v^{\,t}(y) \qquad \textit{for all} \quad s \in R. \tag{5.10}$$

Acknowledgment

The author is indebted to Professor J. P. LaSalle for his remarks. This work was supported by a grant from the National Science Foundation.

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